THE DUALITY BETWEEN κ-POINCARÉ ALGEBRA AND κ-POINCARÉ GROUP

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Abstract. The full duality between the κ-Poincaré algebra and κ-Poincaré group is proved.

1. Introduction

Recently, much attention has been paid to the specific deformation of Poincaré algebra — the so-called κ-Poincaré algebra [1]. Its global counterpart, the κ-Poincaré group, has been obtained by S. Zakrzewski [2]. His method consisted in quantising the Poisson structure on classical r-matrix obtained from $\frac{1}{\kappa}$ expansion of algebra coproduct. Zakrzewski method gives, in principle, the duality group $\Leftrightarrow$ algebra only in the lowest, $\frac{1}{\kappa}$-approximation. However, due to the lack of ordering ambiguities in the quantisation procedure it seemed likely there is a full duality between κ-Poincaré group and algebra. Indeed, it was shown ([3]. [4], [5]) that this is the case in two dimensions. The proof given in the last paper relies heavily on the bicrossproduct structure of κ-Poincaré algebra and group discovered in [5] and [6]. Here, we briefly sketch how to extend this proof to four dimensions. In fact, it is easily seen that the proof presented below works in any dimensions. The full version will appear elsewhere.

II. The κ-Poincaré algebra and κ-Poincaré group

The κ-Poincaré algebra $\mathcal{P}_k$ is defined by the following rules ([1], [6]) ($M_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}$, $N_i = M_{i0}$):

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, && \\
&[M_i, M_j] = i\varepsilon_{ijk}M_k, && [M_i, P_0] = 0, \\
&[M_i, N_j] = i\varepsilon_{ijk}N_k, && [M_i, P_j] = i\varepsilon_{ijk}P_k, \\
&[N_i, N_j] = -i\varepsilon_{ijk}M_k, && [N_i, P_0] = iP_i,
\end{aligned}
$$

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$$[N_i, P_j] = i\delta_{ij}\left(\frac{\kappa}{2}\left(1 - e^{-\frac{2P_0}{\kappa}}\right) + \frac{1}{2\kappa}\vec{P}^{\,2}\right) - \frac{i}{\kappa}P_iP_j, \tag{1}$$

$$\begin{aligned}
&\Delta P_0 = P_0 \otimes I + I \otimes P_0, \qquad \Delta P_i = P_i \otimes e^{-\frac{P_0}{\kappa}} + I \otimes P_i,\\
&M_i = M_i \otimes I + I \otimes M_i,\\
&\Delta N_i = N_i \otimes e^{-\frac{P_0}{\kappa}} + I \otimes N_i - \frac{1}{\kappa}\varepsilon_{ijk}M_j \otimes P_k,\\
&S(P_\mu) = -P_\mu, \qquad S(M_i) = -M_i, \qquad S(N_i) = -N_i + \frac{3i}{2\kappa}P_i,\\
&\varepsilon(X) = 0, \qquad X = P_\mu, M_i, N_i.
\end{aligned}$$

It has ([5], [6]) a natural bicrossproduct ([7]) structure

$$\mathcal{P}_k = T \blacktriangleright\!\!\triangleleft\, U(s0(3,1)). \tag{2}$$

Indeed, it is sufficient to define

$$\begin{aligned}
&M_i \triangleright P_0 = 0, \qquad M_i \triangleright P_k = i\varepsilon_{ijk}P_j,\\
&N_i \triangleright P_0 = iP_i, \qquad N_i \triangleright P_j = i\delta_{ij}\left(\frac{\kappa}{2}\left(1 - e^{-\frac{2P_0}{\kappa}}\right) + \frac{1}{2\kappa}\vec{P}^{\,2}\right) - \frac{i}{\kappa}P_iP_j,\\
&\delta(M_i) = M_i \otimes I, \quad \delta(N_i) = N_i \otimes e^{-\frac{P_0}{\kappa}} - \frac{1}{\kappa}\varepsilon_{ijk}M_j \otimes P_k
\end{aligned} \tag{3}$$

here (N_i, M_i) is the standard Lorentz algebra (with standard coproduct) while T is defined as an algebra generated by P_μ, $\mu = 0, \ldots, 3$ obeying the following relations:

$$\begin{aligned}
&[P_\mu, P_\nu] = 0,\\
&\Delta P_0 = P_0 \otimes I + I \otimes P_0, \quad \Delta P_i = P_i \otimes e^{-\frac{P_0}{\kappa}} + I \otimes P_i, S(P_\mu) = -P_\mu,\\
&\varepsilon(P_\mu) = 0.
\end{aligned} \tag{4}$$

II.2. The κ-Poincaré group. The κ-Poincaré group $\widetilde{\mathcal{P}}_\kappa$ is defined by the following relations ([2], [5]):

$$\begin{aligned}
&[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu),\\
&[\Lambda^\mu{}_\nu, x^\rho] = -\frac{i}{\kappa}((\Lambda^\mu{}_0 - \delta^\mu_0)\Lambda^\rho{}_\nu + (\Lambda^0{}_\nu - \delta^0_\nu)q^{\mu\rho}),\\
&[\Lambda^\mu{}_\nu, \Lambda^\alpha{}_\beta] = 0,\\
&\Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu, \qquad \Delta(x^\mu) = \Lambda^\mu{}_\alpha \otimes x^\alpha + x^\mu \otimes I,\\
&S(\Lambda^\mu{}_\nu) = \Lambda_\nu{}^\mu, \qquad S(x^\mu) = -\Lambda_\nu{}^\mu x^\nu,\\
&\varepsilon(x^\mu) = 0, \qquad \varepsilon(\Lambda^\mu{}_\nu) = \delta^\mu_\nu.
\end{aligned} \tag{5}$$

Again, $\widetilde{\mathcal{P}}_\kappa$ can be defined as a bicrossproduct ([5]):

$$\widetilde{\mathcal{P}}_\kappa = T^* \triangleright\!\!\blacktriangleleft\, C(S0(3,1)).$$

To see this it is sufficient to define ([5]):

$$\beta(x^\mu) = \Lambda^\mu{}_\nu \otimes x^\nu.$$

$$\Lambda^\mu{}_\nu \triangleleft x^\rho = -\frac{i}{\kappa}((\Lambda^\mu{}_0 - \delta^\mu_0)\Lambda^\rho{}_\nu + (\Lambda^0{}_\nu - \delta^0_\nu)g^{\mu\rho}) = [\Lambda^\mu{}_\nu, x^\rho].$$

Moreover, while $C(S0(3,1))$ is the standard algebra of functions defined over Lorentz group while T^* is defined by the following relations:

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu),$$

$$\Delta(x^\mu) = x^\mu \otimes I + I \otimes x^\mu,$$

$$S(x^\mu) = -x^\mu, \qquad \varepsilon(x^\mu) = 0.$$

II.3. Duality. We shall define the dualities:

$$C(S0(3,1)) \Longleftrightarrow U(s0(3,1)),$$

$$T^* \Longleftrightarrow T.$$

First, we define standard duality between the Lorentz group and algebra as follows:

$$\begin{aligned}\langle \Lambda^\mu{}_\nu, M_{\alpha\beta}\rangle &\equiv i\frac{d}{dt}(e^{itM_{\alpha\beta}})^\mu{}_\nu\Big|_{t=0} = (M_{\alpha\beta})^\mu_\nu \\ &= i(\delta^\mu_\alpha g_{\nu\beta} - \delta^\mu_\beta g_{\nu\alpha}).\end{aligned} \tag{6}$$

The following lemma is obvious.

Lemma 1.

$$\langle \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n}, M_{\alpha\beta}\rangle = i\sum_{k=1}^{n}(\delta^{\mu_k}_\alpha g_{\nu_k\beta} - \delta^{\mu_k}_\beta g_{\nu_k\alpha})\prod_{l\neq k}\delta^{\mu_l}_{\nu_l}. \tag{7}$$

The duality $T^* \Leftrightarrow T$ is defined by

$$\langle x^\mu, p_\nu\rangle = i\delta^\mu_\nu.$$

This duality can be fully described as follows. For any function $\psi(x^\mu)$ we define the normal product $:\psi(x^\mu):$ as the one in which all x^0 factors stand leftmost. We then have

Lemma 2 ([6])**.**

$$\langle : F(x^\mu) :, f(p_\nu)\rangle = f\left(i\frac{\partial}{\partial x^\nu}\right)F(x^\mu)\Big|_{x=0}. \tag{8}$$

A simple proof is based on Leibnitz rule and the identity

$$(x^n)^l(x^0)^n = \left(x^0 - i\frac{l}{\kappa}\right)^n(x^m)^l. \tag{9}$$

II.3. The structure of $\triangleleft$ and β operations. In the sequel we shall need some more detailed information concerning the structure of the operations $\triangleleft$ and β. We have the following lemma.

Lemma 3.

$$(\Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n})\triangleleft x^{\rho_1}\triangleleft\cdots\triangleleft x^{\rho_m}=[\cdots[\Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n},\,x^{\rho_1}],\ldots,x^{\rho_m}]. \tag{10}$$

The proof is based on the following rule:

$$(ab)\triangleleft h=(a\triangleleft h_{(1)})(b\triangleleft h_{(2)}). \tag{11}$$

It is slightly more difficult to describe the structure of β operation. To this end we define the τ-operation, acting on an arbitrary product of Λ's and x's as follows: using commutation rules (5), we transpose all x's to the left and then put $x^\mu=0$. By linearity we extend τ to any polynomial in x's and Λ's. The τ-operation has the following obvious property

$$\tau(\tau(P_1)P_2)=\tau(P_1P_2). \tag{12}$$

Lemma 4.

$$\beta(x^{\mu_1}\cdots x^{\mu_n})=(\tau\otimes\mathrm{id})\left(\prod_{k=1}^{n}(\Lambda^{\mu_k}{}_{\nu_k}\otimes x^{\nu_k}+x^{\mu_k}\otimes I)\right). \tag{13}$$

The inductive proof is based on identity (12) and the product rule ([5], [6], [7]):

$$\beta(hg)=(h^{\bar{1}}\triangleleft g_{(1)})g^{\bar{1}}_{(2)}\otimes h^{\bar{2}}g^{\bar{2}}_{(2)}. \tag{14}$$

III. The proof of duality

We have to prove the following duality relations:

$$\langle X,M_{\alpha\beta}\triangleright P_\gamma\rangle=\langle\beta(X),M_{\alpha\beta}\otimes P_\gamma\rangle, \tag{15a}$$

$$\langle\Lambda\triangleleft X,M_{\alpha\beta}\rangle=\langle\Lambda\otimes X,\delta(M_{\alpha\beta})\rangle, \tag{15b}$$

here X is an arbitrary product of x's while Λ is an arbitrary product of Λ's. We assume that the operations $\triangleleft$ and β are known and use relations (15) to prove the structure of $\triangleright$ and δ

Theorem 1. *The following rules are implied by (15a)*

$$\begin{aligned}&M_i\triangleright P_0=0,\qquad M_i\triangleright P_j=i\varepsilon_{ijk}P_k,\qquad N_i\triangleright P_0=iP_i,\\&N_i\triangleright P_j=i\delta_{ij}\left(\frac{\kappa}{2}\left(1-e^{-\frac{2P_0}{\kappa}}\right)+\frac{1}{2\kappa}\vec{P}^{\,2}\right)-\frac{i}{\kappa}P_iP_j.\end{aligned} \tag{16}$$

As an example we shall prove the most complicated last equality. It follows immediately from (8) and the following lemma

Lemma 5.
(i)

$$\langle \beta(x^{m_1}\cdots x^{m_r}), M_{i0}\otimes P_k\rangle = \begin{cases} 0, & r\neq 2,\\ \dfrac{i}{\kappa}(\delta_{m_1k}\delta_{m_2i}+\delta_{m_1i}\delta_{m_2k}-\delta_{m_1m_2}\delta_{ik}), & r=2.\end{cases}$$

(ii) $$\langle \beta((x^0)^n x^{m_1}\cdots x^{m_r}), M_{i0}\otimes P_k\rangle = \begin{cases} 0, & r>0,\\ -\delta_{ik}\left(-\dfrac{2i}{\kappa}\right)^{n-1}, & r=0.\end{cases}$$

Proof. We prove, for example, the first part of the lemma. First, we have

$$\begin{aligned}\langle \beta(x^m), M_{i0}\otimes P_k\rangle &= \langle \Lambda^m{}_\nu \otimes x^\nu, M_{i0}\otimes P_k\rangle\\ &= i\langle \Lambda^m{}_k, M_{i0}\rangle\end{aligned}$$

and by Lemma 4

$$\begin{aligned}\langle \beta(x^m x^n), M_{i0}\otimes P_k\rangle &= \langle \Lambda^m{}_\mu \Lambda^n{}_\nu \otimes x^\mu x^\nu, M_{i0}\otimes P_k\rangle\\ &\quad + \langle [\Lambda^m{}_\mu, x^n]\otimes x^\mu, M_{i0}\otimes P_k\rangle.\end{aligned}$$

The second term on the right-hand side can be evaluated immediately using Lemmas 1 and 2. In order to evaluate the first one let us notice that, by Lemma 2 and the commutation rule for x's the only term that gives a nonvanishing contribution corresponds to $\mu = k$, $\nu = 0$. Let us now consider the case $r > 2$. First, note that

$$\langle x^{\mu_1}\cdots x^{\mu_n}, P_k\rangle = \left(-\frac{i}{\kappa}\right)^{n-1}\delta_{\mu_1 k}\prod_{l=2}^{n}\delta_{\mu_l 0}. \tag{17}$$

Therefore, we have by (17) and the definition of τ

$$\begin{aligned}&\langle \beta(x^{m_1}\cdots x^{m_r}), M_{i0}\otimes P_k\rangle\\ &\quad = \Big\langle (\tau\otimes \mathrm{id})\Big(\prod_{l=1}^{r}(\Lambda^{m_l}{}_{\nu_l}\otimes x^{\nu_l} + x^{m_l}\otimes I)\Big), M_{i0}\otimes P_k\Big\rangle\\ &\quad = \Big\langle (\tau\otimes \mathrm{id})\Big((\Lambda^{m_1}{}_k\otimes x^k)\prod_{l=2}^{r}(\Lambda^{m_l}{}_0\otimes x^0 + x^{m_l}\otimes I)\Big), M_{i0}\otimes P_k\Big\rangle.\end{aligned} \tag{18}$$

We shall prove that for $r\geq 3$

$$(\tau\otimes \mathrm{id})\left((\Lambda^{m_1}{}_k\otimes x^k)\prod_{l=2}^{r}(\Lambda^{m_l}{}_0\otimes x^0 + x^{m_l}\otimes I)\right) = \Lambda_A^{m_1\cdots m_r}\otimes x^A \tag{19}$$

where, for any multiindex A, $\Lambda_A^{m_1\cdots m_r}$ can be decomposed into the sum of monomials, each containing $\Lambda_0^0 - 1$ or/and $\Lambda^m{}_0\Lambda^n{}_0$ or/and $\Lambda^0{}_m\Lambda^0{}_n$ or/and $\Lambda^0{}_m\Lambda^n{}_0$

In order to prove this we use induction with respect to r. We have by (12)

$$
\begin{aligned}
&(\tau\otimes\mathrm{id})\Big((\Lambda^{m_1}{}_k\otimes x^k)\prod_{l=2}^{r+1}(\Lambda^{m_l}{}_0\otimes x^0+x^{m_l}\otimes I)\Big)=(\tau\otimes\mathrm{id})\\
&\quad\cdot\Bigg(\Big((\tau\otimes\mathrm{id})(\Lambda^{m_1}{}_k\otimes x^k)\prod_{l=2}^{r}(\Lambda^{m_l}{}_0\otimes x^0+x^0\otimes I)\Big)(\Lambda_0^{m_{r+1}}\otimes x^0+x^{m_{r+1}}\otimes I)\Bigg)\\
&\quad=(\tau\otimes\mathrm{id})((\Lambda_A^{m_1\cdots m_r}\otimes x^A)(\Lambda_0^{m_{r+1}}\otimes x^0+x^{m_{r+1}}\otimes I))\\
&\quad=\Lambda_A^{m_1\cdots m_r}\Lambda_0^{m_{r+1}}\otimes x^Ax^0+[\Lambda_A^{m_1\cdots m_r},x^{m_{r+1}}]\otimes x^A.
\end{aligned}
$$

The first term on the right-hand side has already the proper structure. In order to prove the same for the second term it is sufficient to use the following commutation rules:

$$
\begin{aligned}
&[\Lambda^0{}_0-1,x^n]=-\frac{i}{\kappa}(\Lambda^0{}_0-1)\Lambda^n{}_0,\\
&[\Lambda^0{}_k,x^n]=-\frac{i}{\kappa}(\Lambda^0{}_0-1)\Lambda^n{}_k,\\
&[\Lambda^k{}_0,x^n]=-\frac{i}{\kappa}(\Lambda^n{}_0\Lambda^k{}_0+g^{nk}(\Lambda^0{}_0-1)).
\end{aligned}
\tag{20}
$$

For $r=3$ relation (15) is verified by simple straightforward calculation.

In order to complete the proof of Lemma (5i) we note that, by Lemma 1,

$$\langle\Lambda,M_{\alpha\beta}\rangle=0$$

if Λ is any monomial containing $\Lambda^0{}_0-1$ or/and $\Lambda^m{}_0\Lambda^n{}_0$ or/and $\Lambda^0{}_m\Lambda^0{}_n$ or/and $\Lambda^0{}_m\Lambda^n{}_0$.

Relation (ii) can be proved along the same lines.

Theorem 2. *The following rules are implied by (15b):*

$$
\begin{aligned}
&\delta(M_i)=M_i\otimes I,\\
&\delta(N_i)=N_i\otimes e^{-\frac{P_0}{\kappa}}-\frac{1}{\kappa}\varepsilon_{ijk}M_j\otimes P_k.
\end{aligned}
$$

Again, as an example, we prove the last equality. It follows from (8) and the following

Lemma 6.

(i)

$$\Big\langle\prod_a\Lambda^{\mu_a}{}_{\nu_a}\triangleleft(x^0)^n,M_{i0}\Big\rangle=\Big(-\frac{i}{\kappa}\Big)^n\Big\langle\prod_a\Lambda^{\mu_a}{}_{\nu_a},M_{i0}\Big\rangle,\tag{21}$$

(ii) *for $r>0$*

$$\Big\langle\prod_a\Lambda^{\mu_a}{}_{\nu_a}\triangleleft(x^0)^l\prod_{p=1}^r x^{m_p},M_{i0}\Big\rangle=\Big(-\frac{i}{\kappa}\Big)\delta_{l0}\delta_{r1}\Big\langle\prod_a\Lambda^{\mu_a}{}_{\nu_a},M_{m_1i}\Big\rangle.\tag{22}$$

We shall prove (i). We use induction with respect to n. First, note the following important property of the commutator $[\Lambda^\mu{}_\nu, x^\rho]$

$$[\Lambda^\mu{}_\nu, x^\rho]\Big|_{\Lambda^\alpha{}_\beta \to \delta^\alpha_\beta} = 0. \tag{23}$$

Therefore, by Lemma 1, we have

$$\Big\langle \prod_a \Lambda^{\mu_a}{}_{\nu_a} \triangleleft x^0, M_{i0} \Big\rangle = \Big\langle \Big[\prod_a \Lambda^{\mu_a}{}_{\nu_a}, x^0 \Big], M_{i0} \Big\rangle = \sum_a \prod_{b \neq a} \delta^{\mu_b}_{\nu_b} \langle [\Lambda^{\mu_a}{}_{\nu_a}, x^0], M_{i0} \rangle.$$

But, by straightforward calculation

$$\langle [\Lambda^{\mu_a}{}_{\nu_a}, x^0], M_{i0} \rangle = \Big(-\frac{i}{\kappa} \Big) \langle \Lambda^{\mu_a}{}_{\nu_a}, M_{i0} \rangle$$

so, appealing again to Lemma 1, we get equality (21) for $n = 1$.

For $n > 1$ we use again (23) and Lemma 3 to infer

$$\Big\langle \prod_a \Lambda^{\mu_a}{}_{\nu_a} \triangleleft (x^0)^n, M_{i0} \Big\rangle = \sum_a \prod_{b \neq a} \delta^{\mu_b}_{\nu_b} \langle \Lambda^{\mu_a}{}_{\nu_a} \triangleleft (x^0)^n, M_{i0} \rangle.$$

By induction hypothesis

$$\begin{aligned} \langle \Lambda^{\mu_a}{}_{\nu_a} \triangleleft (x^0)^{n+1}, M_{i0} \rangle &= \langle [\Lambda^{\mu_a}{}_{\nu_a} x^0] \triangleleft (x^0)^n, M_{i0} \rangle \\ &= \Big(-\frac{i}{\kappa} \Big)^n \langle [\Lambda^{\mu_a}{}_{\nu_a}, x^0], M_{i0} \rangle = \Big(-\frac{i}{\kappa} \Big)^{n+1} \langle \Lambda^{\mu_a}{}_{\nu_a}, M_{i0} \rangle \end{aligned}$$

which gives (i); (ii) can be proved in a similar way.

Now, the full duality follows from the general theory of bicrossproducts.

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